Supplemental: An Approximate Mie Scattering Function for Fog and Cloud Rendering

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1 OUTLINE

In this supplemental we provide a number of additional results:

- section 2 provides full details of the importance sampling of our fog phase function, which includes new sampling methods for the Draine and Cornette-Shanks phase functions.
- section 3 provides additional parametric fitting functions for particle diameters spanning $0 < d < 50\mu\text{m}$, together with plots illustrating the accuracy of these approximations.

2 IMPORTANCE SAMPLING

Our fog phase function is the sum of two non-negative lobes,

$$\phi_{\text{fog}}(\theta) = (1 - w_{\text{D}}) \cdot \phi_{\text{HG}}(\theta) + w_{\text{D}} \cdot \phi_{\alpha,g_{\text{D}}}(\theta). \tag{1}$$

which we sample in two steps. We first select one of the two lobes using mixture weight w_{D} and then sample the selected lobe using the procedures listed below. Since both lobes in our model are themselves normalized phase functions, the Draine lobe is sampled when $\xi_{\text{s}} < w_{\text{D}}$. In practice, the random number $\xi_{\text{s}} \in [0, 1)$ used for lobe selection can be rescaled and used again during lobe sampling.

2.1 Henyey-Greenstein

Using notation $u = \cos\theta$, the HG phase function is [Henyey and Greenstein 1941]

$$\phi_{HG}(u) = \frac{1}{4\pi}\frac{1 - g^2}{(1 + g^2 - 2gu)^{3/2}}, \tag{2}$$

and can be sampled via [Toublanc 1996; Witt 1977]

$$t = \frac{1 - g^2}{1 - g + 2g\xi},$$
$$u = \frac{1 + g^2 - t}{2g} \tag{3}$$

for a random number $\xi \in [0, 1)$. If $g = 0$, or very small, it is better to use isotropic sampling as a fallback, simplifying to

$$u = 2\xi - 1. \tag{4}$$

2.2 Draine

Draine's phase function is a two-parameter distribution,

$$\phi_{\alpha,g}(\theta) = \frac{1}{4\pi}\frac{1 - g^2}{(1 + g^2 - 2g\cos\theta)^{3/2}}\frac{1 + \alpha\cos^2\theta}{1 + \alpha(1 + 2g^2)/3}, \tag{5}$$

which generalizes the Cornette-Shanks (CS) phase function [Cornette and Shanks 1992]. The original CS model is recovered when $\alpha = 1$. It has been claimed that CS cannot be sampled using CDF inversion [Toublanc 1996]. On the contrary, we found (using Mathematica) that the CDF for Draine cosines $u = \cos\theta$ is

$$\xi = \frac{W(u) + 2\alpha g^5\left(V(u) + u\right) + (\alpha + 3)g^3 V(u) - 2\alpha\left(V(u) - 1\right) - 2\alpha g u}{2g^3\left(2\alpha g^2 + \alpha + 3\right)V(u)}$$
$$W(u) = -2\alpha g^6 + g^4\left(\alpha\left(u^2 - 2\right) - 3\right) - g^2\left(\alpha\left(V(u) + u^2 - 2\right) + 3\left(V(u) - 1\right)\right)$$

where $V(u) = \sqrt{g^2 - 2gu + 1}$ and ξ is a random number drawn uniformly from $[0, 1)$. After a change of variable $u \to (1 + g^2 - v^2)/(2g)$ to remove the square root, v follows from solution of a quartic equation, which produces u using the above mapping. After factorization of common terms, we find the following form:

$$t_0 = \alpha - \alpha g^2$$
$$t_1 = \alpha g^4 - \alpha$$
$$t_2 = -3 \cdot (4(g^4 - g^2) + t_1 \cdot (1 + g^2))$$
$$t_3 = g \cdot (2\xi - 1)$$
$$t_4 = 3g^2 \cdot (1 + t_3) + \alpha \cdot (2 + g^2 \cdot (1 + (1 + 2g^2) \cdot t_3))$$
$$t_5 = t_0 \cdot (t_1 t_2 + t_4^2) + t_1^3$$
$$t_6 = t_0 \cdot 4(g^4 - g^2)$$
$$t_7 = \left(t_5 + \sqrt{t_5^2 - t_6^3}\right)^{\frac{1}{3}}$$
$$t_8 = 2\frac{t_1 + t_6/t_7 + t_7}{t_0}$$
$$t_9 = \sqrt{6(1 + g^2) + t_8}$$
$$u = \frac{g}{2} + \left(\frac{1}{2g} - \frac{1}{8g}\left(\sqrt{6(1 + g^2) - t_8 + 8t_4/(t_0 t_9)} - t_9\right)^2\right), \tag{6}$$

which exactly samples the Draine (and therefore CS) phase function. A Mathematica implementation is provided with the supplemental material for additional validation.

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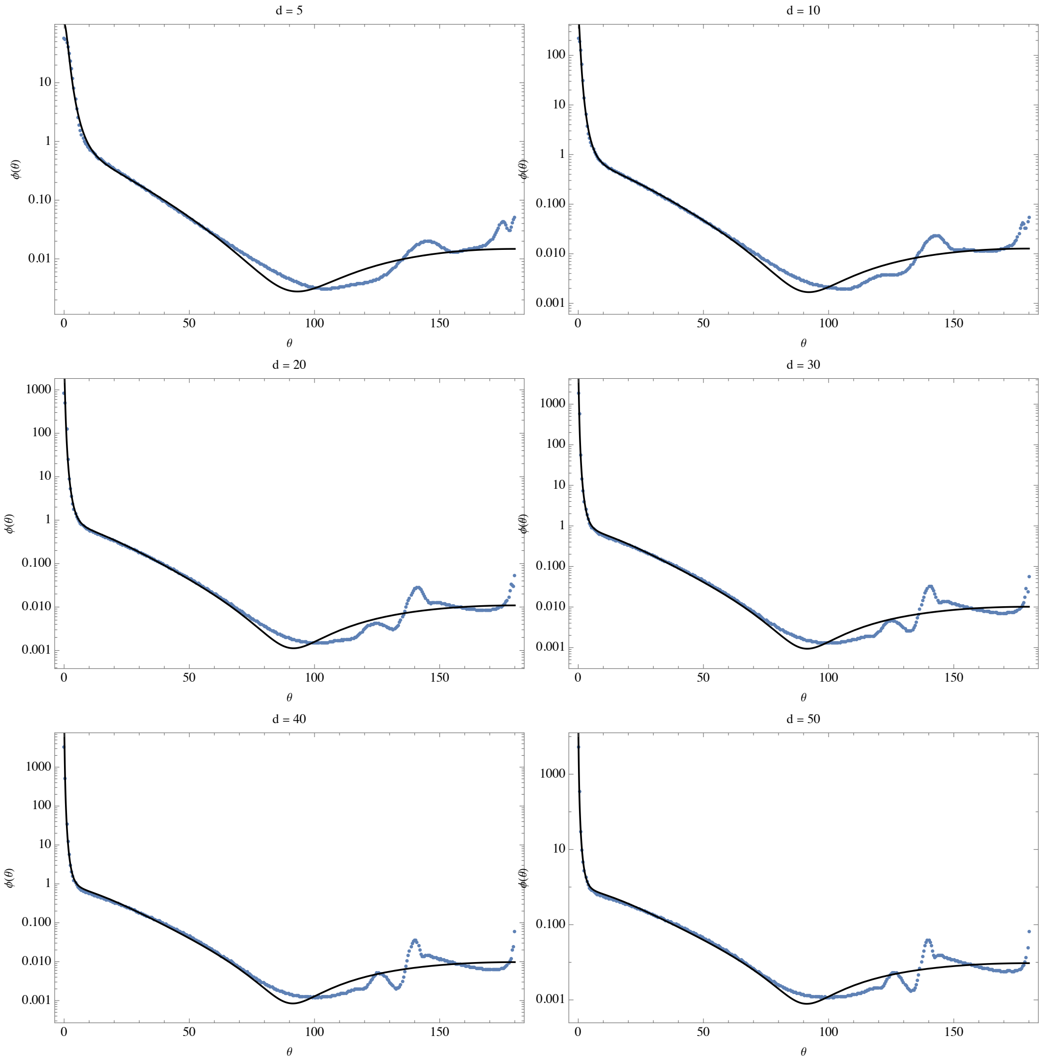

Figure 1: Comparison of our large-particle parametric fit (black) to reference (dots).

3 PARAMETRIC FITS

In this section we are comparing plots of our model against the tabulated Mie reference and propose extensions for the range $d \in [0, 5]$µm.

3.1 Large particles, Diameter $5\,\mu\mathrm{m} \leq d \leq 50\,\mu\mathrm{m}$

Figure 1 compares our fitted parametric model for large particles,

$$g_{\mathrm{HG}}(d) = \mathrm{e}^{-\frac{0.0990567}{d-1.67154}}, \tag{7}$$

$$g_{\mathrm{D}}(d) = \mathrm{e}^{-\frac{2.20679}{d+3.91029}-0.428934}, \tag{8}$$

$$\alpha(d) = \mathrm{e}^{3.62489-\frac{8.29288}{d+5.52825}}, \tag{9}$$

$$w_{\mathrm{D}}(d) = \mathrm{e}^{-\frac{0.599085}{d-0.641583}-0.665888} \tag{10}$$

(repeated here from the main abstract) to reference data for a variety of particle diameters.

3.2 Small particles, Diameter $d <= 0.1\,\mu m$

For small diameter ($d < 0.1\,\mu m$) we found the following parametric fit to be very accurate:

$$g_{HG}(d) = 13.8d^2, \tag{11}$$
$$g_D(d) = 1.1456\, d \sin(9.29044d), \tag{12}$$
$$\alpha(d) = 250, \tag{13}$$
$$w_D(d) = 0.252977 - 312.983d^{4.3}. \tag{14}$$

Figure 2 shows a comparison of this model to reference data for a variety of small diameters. In the limit of vanishing particle diameter ($d \to 0$) the phase function approaches Rayleigh scattering, which can be represented in two distinct ways with our model: one that uses the Draine term exclusively: $w_D = 1, \alpha = 1, g_{HG} = 0, g_D = 0$, and one that mixes Draine with HG: $w_D = 0.25, \alpha = \infty, g_{HG} = 0, g_D = 0$. We found that the latter representation served as a more accurate limit for small particle size (when d is not quite zero), and therefore built the above approximation with $\alpha = 250$. This avoids having to special-case the sampling routine for $\alpha = \infty$, and results in negligible error.

3.3 Mid-range particles, Diameter $0.1\,\mu m < d < 1.5\,\mu m$

Particle diameters d in the range $[0.1, 5]$ presented more of a challenge for our two-lobe parametric model. For this range, we split the fitting into two subranges. For particles with diameters $0.1 < d < 1.5$ we propose

$$g_{HG}(d) = 0.862 - 0.143 \log^2(d), \tag{15}$$
$$g_D(d) = 0.379685 \cos\left(1.19692 \cos\left(\frac{(\log(d) - 0.238604)(\log(d) + 1.00667)}{0.507522 - 0.15677 \log(d)}\right) + 1.37932 \log(d) + 0.0625835\right) + 0.344213, \tag{16}$$
$$\alpha(d) = 250, \tag{17}$$
$$w_D(d) = 0.146209 \cos(3.38707 \log(d) + 2.11193) + 0.316072 + 0.0778917 \log(d). \tag{18}$$

The accuracy of this fit is illustrated in Figure 3.

3.4 Mid-range particles, Diameter $1.5\,\mu m <= d < 5\,\mu m$

For particles with diameters $1.5 <= d < 5$ we propose

$$g_{HG}(d) = 0.0604931 \log(\log(d)) + 0.940256, \tag{19}$$
$$g_D(d) = 0.500411 - \frac{0.081287}{-2\log(d) + \tan(\log(d)) + 1.27551}, \tag{20}$$
$$\alpha(d) = 7.30354 \log(d) + 6.31675, \tag{21}$$
$$w_D(d) = 0.026914(\log(d) - \cos(5.68947(\log(\log(d)) - 0.0292149))) + 0.376475. \tag{22}$$

The accuracy of this fit is illustrated in Figure 4.

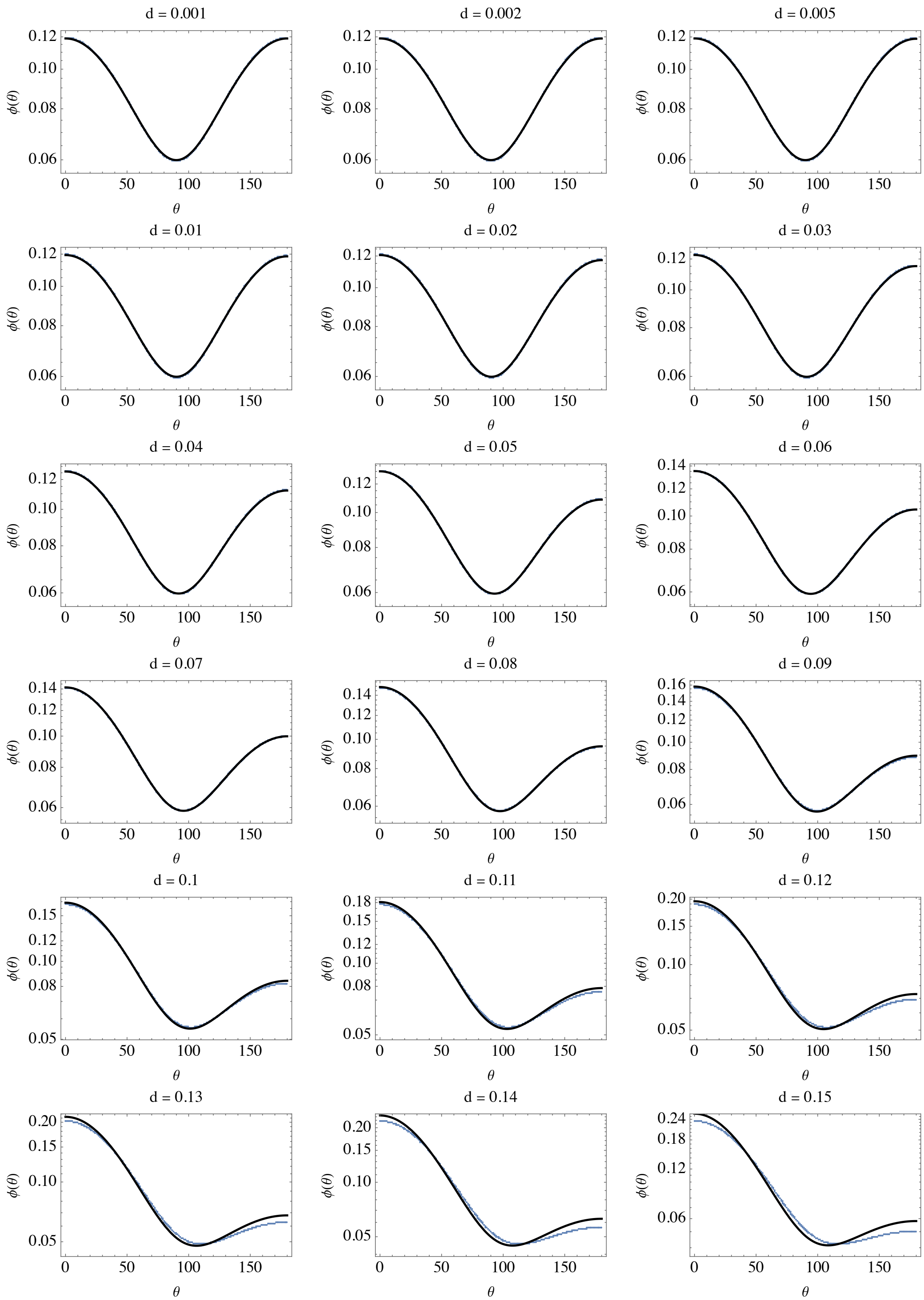

Figure 2: Comparison of our small-particle parametric fit (black, Eq. 11) to reference (dots).

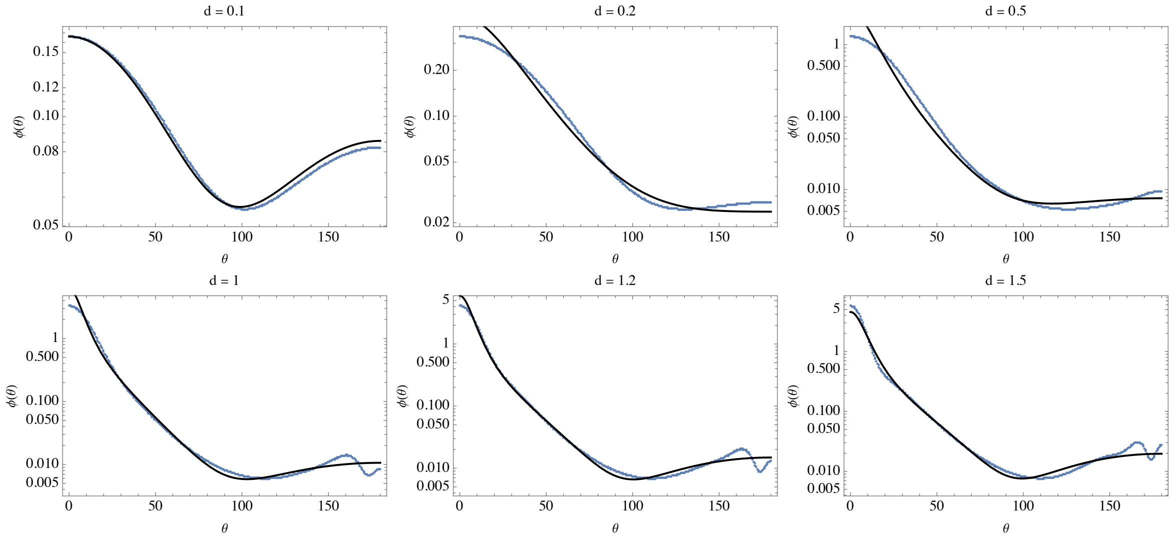

Figure 3: Comparison of our first medium-range-diameter parametric fit (black, Eq. 15) to reference (dots).

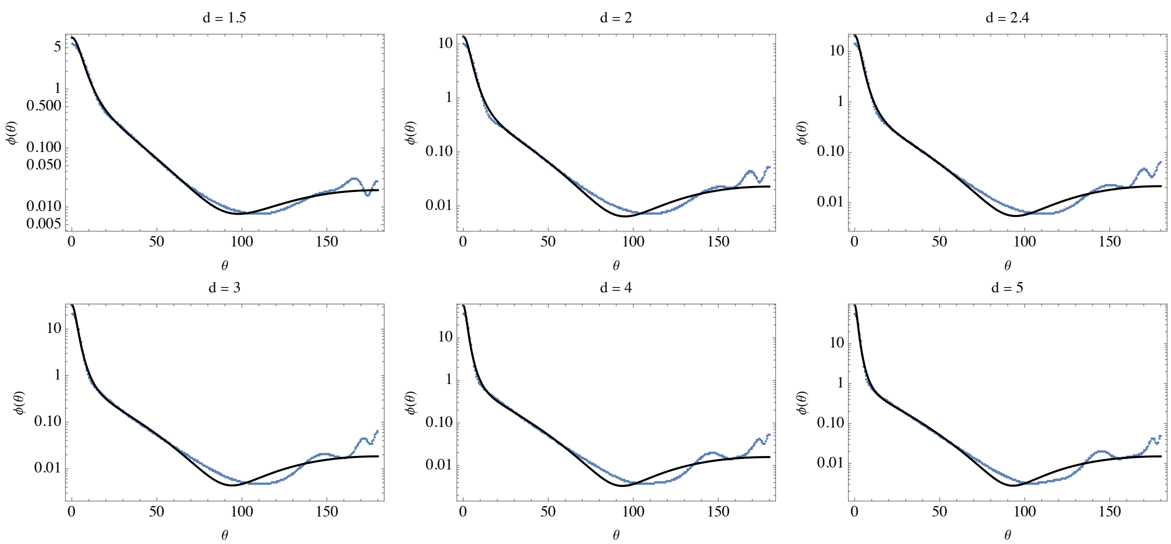

Figure 4: Comparison of our second medium-range-diameter parametric fit (black, Eq. 19) to reference (dots).

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